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BINOMIAL SUMS, MOMENTS AND INVARIANT SUBSPACES

BY

ISABELLE CHALENDAR

Institut Girard Desargues, Université Lyon 1
Boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France
e-mail: chalenda@desargues.univ-lyon1.fr

AND

KARIM KELLAY AND THOMAS RANSFORD*

Département de mathématiques et de statistique, Université Laval
Québec (Québec), Canada G1K 7P4
e-mail: kellay@mat.ulaval.ca e-mail: ransford@mat.ulaval.ca

ABSTRACT

The main result of this paper is that if a sequence of complex numbers $(a_n)_{n\geq 0}$ satisfies

$$\sum_{\substack{k=0\\ k \text{ even}}}^{n} \binom{n}{k} a_k = O(n^r) \quad \text{and} \quad \sum_{\substack{k=0\\ k \text{ odd}}}^{n} \binom{n}{k} a_k = O(n^r) \quad \text{as } n \to \infty,$$

for some integer $r \geq 0$, then $a_n = 0$ for all $n > r$. As an application, we deduce a localized form of a theorem of Allan about nilpotent elements in Banach algebras, and this in turn leads to an invariant-subspace theorem. As a further application, we prove a variant of Carleman's theorem on the unique determination of probability distributions by their moments. The paper concludes with a quantitative form of the main result.

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1. Introduction

Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers, and let $r \geq 0$. Clearly, if $a_n = 0$ for all $n > r$, then

$$\sum_{k=0}^{n} \binom{n}{k} a_k = O(n^r) \quad \text{as } n \to \infty.$$

The converse is false. For example, the sequence $a_n = (-1)^n$ satisfies

$$\sum_{k=0}^{n} \binom{n}{k} a_k = (1 + (-1))^n = 0 \quad \text{for all } n \geq 1,$$

but a_n does not even tend to zero. There is however a partial converse, which, in view of the example above, is perhaps a little surprising.

THEOREM 1.1: *Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers, and let r be a non-negative integer. Assume that*

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} a_k = O(n^r) \quad \text{and} \quad \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} a_k = O(n^r) \quad \text{as } n \to \infty. \tag{1}$$

Then $a_n = 0$ for all $n > r$.

This theorem is the focal point of the paper. We present a short proof of it in §2 using the theory of entire functions of exponential type. Then, in §3, we give an application to Banach algebras, which leads to a theorem on invariant subspaces. A further application, to the determination of probability distributions, is outlined in §4. In §5, we give a second proof of Theorem 1.1 which, though longer, is more 'elementary' than the first, and has the advantage that it yields a quantitative version of the theorem. Finally, in §6, we make some concluding remarks and pose a few questions.

2. Proof of Theorem 1.1

An entire function $f\colon \mathbf{C} \to \mathbf{C}$ is of **exponential type** if

$$\tau := \limsup_{|z|\to\infty} \frac{\log |f(z)|}{|z|} < \infty,$$

in which case τ is called the **type** of f. If further $\tau = 0$, then f is said to be of **minimal exponential type**. We refer to Boas's book [3] for background on entire functions. In particular, we shall have recourse to the following version of the Phragmén–Lindelöf principle.

PHRAGMÉN–LINDELÖF PRINCIPLE ([3, Theorem 6.2.13]): *Let f be an entire function of minimal exponential type, and let $r \geq 0$. Suppose that, on the real axis, $f(x) = O(|x|^r)$ as $x \to \pm\infty$. Then f is a polynomial of degree at most r.*

Proof of Theorem 1.1: Consider the expression

$$a(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n.$$

We shall show successively that this defines an entire function of exponential type, that it is of minimal exponential type and, finally, that it is a polynomial of degree at most r, which will yield the desired conclusion.

For $n \geq 0$, set

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \quad \text{and} \quad c_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_k.$$

The hypotheses on (a_n) ensure that both $b_n, c_n = O(n^r)$ as $n \to \infty$. Hence, if we set

$$b(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n \quad \text{and} \quad c(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n,$$

then b and c are entire functions of exponential type at most 1. Comparing coefficients of z^n in the identities $e^z(e^{-z}b(z)) = b(z)$ and $e^{-z}(e^z c(z)) = c(z)$, we see that both $e^{-z}b(z)$ and $e^z c(z)$ have the same coefficients as $a(z)$, and therefore

$$a(z) = e^{-z} b(z) = e^z c(z) \quad (z \in \mathbf{C}). \tag{2}$$

In particular, a too is an entire function. Further, (2) implies that $a^2 = bc$, and therefore a is also of exponential type at most 1.

We next show that a is in fact of minimal exponential type. To this end, we consider the Laplace transforms A, B, C of a, b, c respectively. Thus, for example,

$$A(\zeta) = \int_0^\infty a(x) e^{-x\zeta}\, dx.$$

Since a, b, c are all of exponential type at most 1, it follows that A, B, C are well-defined and holomorphic in $\{\zeta\colon \operatorname{Re}\zeta > 1\}$. Moreover, for $\operatorname{Re}\zeta > 1$,

$$A(\zeta) = \int_0^\infty \Big(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\Big) e^{-x\zeta}\, dx = \sum_{n=0}^{\infty} \frac{a_n}{n!} \int_0^\infty x^n e^{-x\zeta}\, dx = \sum_{n=0}^{\infty} \frac{a_n}{\zeta^{n+1}}, \tag{3}$$

with analogous expansions for B, C. Using standard properties of Laurent series, it follows that A, B, C extend holomorphically to $\{\zeta\colon |\zeta| > 1\}$. Now, taking Laplace transforms in (2) gives, for $\operatorname{Re}\zeta > 1$,

$$A(\zeta) = B(\zeta+1) = C(\zeta-1).$$

Therefore A in fact extends holomorphically to $\{\zeta\colon |\zeta+1| > 1\} \cup \{\zeta\colon |\zeta-1| > 1\}$, in other words, to $\mathbf{C} \setminus \{0\}$. In conjunction with the Laurent expansion (3), this implies that

$$|a_n|^{1/n} \to 0 \quad \text{as } n \to \infty.$$

It follows that a is of minimal exponential type.

Finally, we show that a is a polynomial. For $x \geq 0$, we have

$$|b(x)| \leq \sum_{n=0}^{\infty} \frac{|b_n|}{n!} x^n \leq \sum_{n=0}^{\infty} \frac{K(n+r)\cdots(n+1)}{n!} x^n = K(x^r e^x)^{(r)},$$

where K is a constant independent of x. Therefore, using (2) again,

$$|a(x)| = |e^{-x} b(x)| \leq K e^{-x} (x^r e^x)^{(r)} = O(x^r) \quad \text{as } x \to \infty.$$

A similar calculation with $c(x)$ in place of $b(x)$ shows that $a(x) = O(|x|^r)$ as $x \to -\infty$. As a is of minimal exponential type, we can apply the Phragmén–Lindelöf principle cited earlier to deduce that a is a polynomial of degree at most r. Hence $a_n = 0$ for all $n > r$, and the proof is complete. ∎

We conclude this section with a 'little o' version of Theorem 1.1. It is actually a simple consequence of the 'big O' form.

Corollary 2.1: *Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers, and let r be a non-negative integer. Assume that*

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} a_k = o(n^r) \quad \text{and} \quad \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} a_k = o(n^r) \quad \text{as } n \to \infty.$$

Then $a_n = 0$ for all $n \geq r$.

Proof: From Theorem 1.1 we already have $a_n = 0$ for all $n > r$, so it only remains to show that $a_r = 0$. Now, for all $n \geq r$, we have

$$\binom{n}{r} a_r = \sum_{k=0}^{n} \binom{n}{k} a_k - \sum_{k=0}^{r-1} \binom{n}{k} a_k.$$

The first term on the right-hand side is $o(n^r)$ by hypothesis, and the second term is even $O(n^{r-1})$ (or $= 0$ if $r = 0$). Therefore the left-hand side is also $o(n^r)$, and this can only happen if $a_r = 0$. ∎

3. Application to Banach algebras and invariant subspaces

Recently, Allan [1] proved the following lemma of Gelfand–Hille type, which he used to give an improved version of a result of Kalton [4] on sums of idempotents.

ALLAN'S LEMMA ([1, Lemma 3]): *Let A be a Banach algebra with identity 1, let $x \in A$, and suppose that, for integers $m \geq 0$, $r \geq 0$,*

$$\left\| x^m\big((1+x)^n - (1-x)^n\big) \right\| = O(n^r) \quad \text{as } n \to \infty. \tag{4}$$

Then $x^{m+r+2} = 0$ if r is odd, while $x^{m+r+1} = 0$ if r is even. Moreover, if r is odd and if

$$\left\| x^m\big((1+x)^n - (1-x)^n\big) \right\| = o(n^r) \quad \text{as } n \to \infty,$$

then $x^{m+r} = 0$.

Theorem 1.1 and Corollary 2.1 permit us to prove a localized form of this result. (Here, and in what follows, E^* denotes the dual space of a Banach space E.)

THEOREM 3.1: *Let A be a Banach algebra with identity 1, let $x \in A$ and let $\varphi \in A^*$. Suppose that, for some integer $r \geq 0$,*

$$\varphi\big((1+x)^n - (1-x)^n\big) = O(n^r) \quad \text{as } n \to \infty. \tag{5}$$

Then $\varphi(x^n) = 0$ for all odd integers $n > r$. If further

$$\varphi\big((1+x)^n - (1-x)^n\big) = o(n^r) \quad \text{as } n \to \infty,$$

then $\varphi(x^n) = 0$ for all odd integers $n \geq r$.

Proof: Observe that

$$\varphi\big((1+x)^n - (1-x)^n\big) = 2 \sum_{\substack{k=0, \\ k \text{ odd}}}^{n} \binom{n}{k} \varphi(x^k).$$

The result therefore follows by applying Theorem 1.1 and Corollary 2.1 to the sequence

$$a_n = \begin{cases} \varphi(x^n), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases} \qquad \blacksquare$$

Allan's lemma is an easy consequence of Theorem 3.1. Indeed, given x, m, r satisfying (4), take $\psi \in A^*$ and define $\varphi \in A^*$ by

$$\varphi(y) = \psi(x^m y) \quad (y \in A).$$

Then (5) holds, so applying Theorem 3.1 we deduce that $\psi(x^{m+n}) = 0$ for all odd integers $n > r$. As ψ is arbitrary, it follows that $x^{m+n} = 0$ for all odd integers $n > r$. This gives the $O(n^r)$ part of the result, and the $o(n^r)$ part is proved similarly.

It is much less clear whether Theorem 3.1 can be deduced from Allan's result. Allan's proof begins by showing that (4) implies that x is quasi-nilpotent, which fact he then exploits via elementary spectral theory and entire function theory to obtain the desired conclusion. However, the weaker hypothesis (5) no longer implies that x is quasi-nilpotent, so a proof of Theorem 3.1 has to proceed along different lines. It was the search for a proof independent of spectral theory which eventually led to a result purely about complex numbers, Theorem 1.1. It should be added, however, that the last part of the proof of Theorem 1.1 (using the Phragmén–Lindelöf principle) owes much to Allan's paper.

Because the proof of Theorem 3.1 uses no spectral theory, it is just as valid for real Banach algebras as for complex ones, and hence so too is the deduction of Allan's lemma. Perhaps more interestingly, Theorem 3.1 also leads to the following invariant-subspace theorem. (We use the usual notation $\langle\cdot,\cdot\rangle$ for duality, and I for the identity operator.)

THEOREM 3.2: *Let E be a (real or complex) infinite-dimensional Banach space, and let T be a bounded linear operator on E. Suppose that there exist a non-zero $\xi_0 \in E$, a non-zero $\psi_0 \in E^*$ and an integer $r \geq 0$ such that*

$$\langle \psi_0, \big((I+T)^n - (I-T)^n\big)\xi_0 \rangle = O(n^r) \quad \text{as } n \to \infty.$$

Then T^2 has a non-trivial closed invariant subspace.

Proof: The result is obvious if $T = 0$, so we can suppose that $T \neq 0$. Further, if T is not injective, then $\ker(T)$ is a non-trivial closed T^2-invariant subspace, and we are done. So we may as well assume from the outset that T is injective.

Let A be the Banach algebra of bounded linear operators on E, let $x = T$ and let $\varphi \in A^*$ be given by

$$\varphi(S) = \langle \psi_0, S\xi_0 \rangle \quad (S \in A).$$

Then the hypothesis on T implies that (5) holds, so we can apply Theorem 3.1 to conclude that $\langle \psi_0, T^n \xi_0 \rangle = 0$ for all odd integers $n > r$. Let M be the closed linear span of $\{T^n\xi_0 \colon n \text{ odd}, n > r\}$. Then M is a closed T^2-invariant subspace of E. Further, $M \neq 0$ since T is injective, and $M \neq E$ since $\psi_0(M) = 0$. Thus M is the required subspace. ∎

There is also a version of Theorem 3.1 in which the central '−' is replaced by '+'.

THEOREM 3.3: *Let A be a Banach algebra with identity 1, let $x \in A$ and let $\varphi \in A^*$. Suppose that, for some integer $r \geq 0$,*

$$\varphi\big((1+x)^n + (1-x)^n\big) = O(n^r) \quad \text{as } n \to \infty.$$

Then $\varphi(x^n) = 0$ for all even integers $n > r$. If further

$$\varphi\big((1+x)^n + (1-x)^n\big) = o(n^r) \quad \text{as } n \to \infty,$$

then $\varphi(x^n) = 0$ for all even integers $n \geq r$.

Proof: This time

$$\varphi\big((1+x)^n + (1-x)^n\big) = 2 \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} \varphi(x^k),$$

so the result follows by applying Theorem 1.1 and Corollary 2.1 to the sequence

$$a_n = \begin{cases} \varphi(x^n), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \qquad \blacksquare$$

Both Allan's lemma and Theorem 3.2 likewise have '+' analogues, which we shall not state here. Finally, putting the '−' and the '+' results together, we obtain an invariant-subspace theorem for T (rather than merely for T^2).

THEOREM 3.4: *Let E be a (real or complex) infinite-dimensional Banach space, and let T be a bounded linear operator on E. Suppose that there exist a non-zero $\xi_0 \in E$, a non-zero $\psi_0 \in E^*$ and an integer $r \geq 0$ such that both*

$$\langle \psi_0, (I+T)^n \xi_0 \rangle = O(n^r) \quad \text{and} \quad \langle \psi_0, (I-T)^n \xi_0 \rangle = O(n^r) \quad \text{as } n \to \infty.$$

Then T has a non-trivial closed invariant subspace.

Proof: This follows the same lines as the proof of Theorem 3.2. As before, we can suppose that T is injective. This time, applying both Theorems 3.1 and 3.3, we have $\langle \psi_0, T^n \xi_0 \rangle = 0$ for *all* integers $n > r$. Let M be the closed linear span of $\{T^n \xi_0 \colon n > r\}$. Then M is now T-invariant and, as before, $M \neq 0$ and $M \neq E$. $\blacksquare$

4. Application to probability distributions

It is well known that a probability distribution on $\mathbf{R}$ is uniquely determined by its moments, provided that they are finite and do not grow too rapidly.

CARLEMAN'S THEOREM (see e.g. [5, p. 126]): *Let μ, ν be Borel probability measures on $\mathbf{R}$ all of whose moments are finite. Suppose that, for each $n \geq 0$,*

$$S_n := \int_{-\infty}^{\infty} t^n \, d\mu(t) = \int_{-\infty}^{\infty} t^n \, d\nu(t),$$

and further, that

$$\sum_{n=1}^{\infty} S_{2n}^{-1/2n} = \infty.$$

Then $\mu = \nu$.

In this section, we prove an analogue of Carleman's theorem for the complex moments $\int_{-\infty}^{\infty} (1+it)^n \, d\mu(t)$, but with the added twist that, even if the moments $\int_{-\infty}^{\infty} (1+it)^n \, d\nu(t)$ are only 'approximately' equal to those of μ, then still $\mu = \nu$. Here is the precise result.

THEOREM 4.1: *Let μ, ν be Borel probability measures on $\mathbf{R}$ all of whose moments are finite. Suppose that, for some integer $r \geq 0$,*

$$Z_n := \int_{-\infty}^{\infty} (1+it)^n \, d\mu(t) = \int_{-\infty}^{\infty} (1+it)^n \, d\nu(t) + O(n^r) \quad \text{as } n \to \infty,$$

and further, that

$$\sum_{n=1}^{\infty} |Z_{2n}|^{-1/2n} = \infty. \tag{6}$$

Then $\mu = \nu$.

The proof of Theorem 4.1, like that of Carleman's theorem, is based on the theory of quasi-analytic classes. We refer to Koosis's book [5, Chapter IV] for background in this subject. In particular, we shall need a slight variant of the Denjoy–Carleman theorem. To state it, we adopt the notation of [5, p. 79]: given a subinterval I (bounded or unbounded) of $\mathbf{R}$, and a sequence $(M_n)_{n \geq 0}$ of positive numbers, we write $C_I(\{M_n\})$ for the family of all C^∞-functions $f\colon I \to \mathbf{C}$ satisfying

$$|f^{(n)}(x)| \leq c_f \rho_f^n M_n \quad (x \in I, \ n \geq 0),$$

where c_f, ρ_f are constants depending on f.

LEMMA 4.2: *Let $(M_n)_{n\geq 0}$ be a sequence of positive numbers satisfying*

$$M_0 = 1, \quad M_n^2 \leq M_{n-1}M_{n+1} \quad (n \geq 1) \quad \text{and} \quad \sum_{n=1}^{\infty} M_n^{-1/n} = \infty. \tag{7}$$

Let $f \in \mathcal{C}_{\mathbf{R}}(\{M_n\})$, and suppose that, for some integer $r \geq 0$,

$$f^{(n)}(0) = 0 \quad \text{for all } n > r.$$

Then f is constant on $\mathbf{R}$.

Proof: Define $g\colon \mathbf{R} \to \mathbf{C}$ by

$$g(x) = f(x) - \sum_{k=0}^{r} \frac{f^{(k)}(0)}{k!} x^k \quad (x \in \mathbf{R}).$$

Then $g \in C^\infty(\mathbf{R})$ and $g^{(n)}(0) = 0$ for all $n \geq 0$.

Now let I be an interval containing 0. Then certainly $f|I \in \mathcal{C}_I(\{M_n\})$. If, further, I is bounded, then $\mathcal{C}_I(\{M_n\})$ contains the polynomials, and hence also $g|I \in \mathcal{C}_I(\{M_n\})$. The standard form of the Denjoy–Carleman theorem (see e.g. [5, p. 97]) now allows us to conclude that $g|I \equiv 0$. As this holds for every bounded interval I containing 0, it follows that $g \equiv 0$ on $\mathbf{R}$.

Thus f is a polynomial. But also f is bounded on $\mathbf{R}$, since, by definition of $\mathcal{C}_{\mathbf{R}}(\{M_n\})$, we have $|f(x)| \leq c_f$ for all x. Therefore, finally, f is constant on $\mathbf{R}$. ∎

Proof of Theorem 4.1: First observe that, for each $n \geq 0$,

$$\int_{-\infty}^{\infty} (1+it)^n \, d\mu(t) - \int_{-\infty}^{\infty} (1+it)^n \, d\nu(t) = \sum_{k=0}^{n} \binom{n}{k} i^k \int_{-\infty}^{\infty} t^k \, d(\mu-\nu)(t).$$

By hypothesis, the left-hand side is $O(n^r)$ as $n \to \infty$. Hence, taking real and imaginary parts of the right-hand side, it follows that

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} \int_{-\infty}^{\infty} t^k \, d(\mu-\nu)(t) = O(n^r) \text{ and } \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} \int_{-\infty}^{\infty} t^k \, d(\mu-\nu)(t) = O(n^r).$$

Applying Theorem 1.1, we deduce that

$$\int_{-\infty}^{\infty} t^n \, d(\mu-\nu)(t) = 0 \quad \text{for all } n > r. \tag{8}$$

The rest of the proof follows the same lines as that of Carleman's theorem. For $n \geq 0$, define

$$M_n = \frac{1}{2}\int_{-\infty}^{\infty} |t|^n \, d(\mu+\nu)(t).$$

We claim that the sequence $(M_n)_{n\geq 0}$ satisfies the conditions (7) of Lemma 4.2. Indeed, that $M_0 = 1$ is clear, and $M_n^2 \leq M_{n-1}M_{n+1}$ for $n \geq 1$ by Hölder's inequality. The verification of the remaining condition $\sum_{n=1}^{\infty} M_n^{-1/n} = \infty$ is a bit more technical, and is postponed to the end of the proof. Assuming this for the moment, define $f\colon \mathbf{R} \to \mathbf{C}$ by

$$f(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-itx} \, d(\mu-\nu)(t) \quad (x \in \mathbf{R}).$$

Then $f \in C^\infty(\mathbf{R})$ and, for each $n \geq 0$,

$$f^{(n)}(x) = \frac{1}{2}\int_{-\infty}^{\infty} (-it)^n e^{-itx} \, d(\mu-\nu)(t) \quad (x \in \mathbf{R}).$$

In particular,

$$|f^{(n)}(x)| \leq \frac{1}{2}\int_{-\infty}^{\infty} |t|^n \, d(\mu+\nu)(t) = M_n \quad (x \in \mathbf{R},\ n \geq 0),$$

so $f \in \mathcal{C}_{\mathbf{R}}(\{M_n\})$. Also, using (6), we have

$$f^{(n)}(0) = \frac{1}{2}\int_{-\infty}^{\infty} (-it)^n \, d(\mu-\nu)(t) = 0 \quad \text{for all } n > r.$$

Applying Lemma 4.2, we deduce that f is constant. Moreover, as μ and ν are both probability measures,

$$f(0) = \frac{1}{2}\int_{-\infty}^{\infty} 1 \, d(\mu-\nu) = \frac{1}{2} - \frac{1}{2} = 0.$$

Thus $f \equiv 0$, or in other words,

$$\int_{-\infty}^{\infty} e^{-itx} \, d\mu(t) = \int_{-\infty}^{\infty} e^{-itx} \, d\nu(t) \quad (x \in \mathbf{R}).$$

Finally, by the uniqueness theorem for Fourier transforms, we conclude that $\mu = \nu$, as desired.

It remains to justify the claim that $\sum_{n=1}^{\infty} M_n^{-1/n} = \infty$. We argue by contradiction. Suppose, if possible, that $\sum_{n=1}^{\infty} M_n^{-1/n} < \infty$. By Hölder's inequality, the sequence $M_n^{-1/n}$ is decreasing, and hence, for each $n \geq 1$,

$$nM_{2n}^{-1/2n} \leq \sum_{k=n+1}^{2n} M_k^{-1/k} \leq \sum_{k=n+1}^{\infty} M_k^{-1/k}.$$

In particular, it follows that

$$nM_{2n}^{-1/2n} \to 0 \quad \text{as } n \to \infty. \tag{9}$$

Now, for each $n \geq 1$,

$$\begin{aligned}
|Z_{2n}| &= \left| \int_{-\infty}^{\infty} (t-i)^{2n}\, d\mu(t) \right| \\
&\geq \int_{|t|\geq 4n} \operatorname{Re}\big((t-i)^{2n}\big)\, d\mu(t) - \int_{|t|<4n} |t-i|^{2n}\, d\mu(t) \\
&\geq \int_{|t|\geq 4n} t^{2n} \operatorname{Re}\big((1-i/t)^{2n}\big)\, d\mu(t) - (5n)^{2n}.
\end{aligned}$$

For $|t| \geq 4n$, we have

$$\big|\arg(1-i/t)^{2n}\big| = \big|2n \arg\big(1-i/t\big)\big| \leq 2n/|t| \leq \frac{1}{2} \leq \frac{\pi}{3},$$

and consequently

$$\operatorname{Re}\big((1-i/t)^{2n}\big) \geq \cos(\pi/3)\big|1-i/t\big|^{2n} \geq \frac{1}{2}\big(1-1/|t|\big)^{2n} \geq \frac{1}{2}\big(1-2n/|t|\big) \geq \frac{1}{4}.$$

Hence

$$|Z_{2n}| \geq \frac{1}{4} \int_{|t|\geq 4n} t^{2n}\, d\mu(t) - (5n)^{2n} \geq \frac{1}{4} \int_{-\infty}^{\infty} t^{2n}\, d\mu(t) - \frac{1}{4}(4n)^{2n} - (5n)^{2n}.$$

Now by (8), if $2n > r$ then

$$\int_{-\infty}^{\infty} t^{2n}\, d\mu(t) = \frac{1}{2} \int_{-\infty}^{\infty} t^{2n}\, d(\mu+\nu)(t) = M_{2n}.$$

Substituting this into the previous inequality, and rearranging, we obtain

$$M_{2n} \leq 4|Z_{2n}| + (4n)^{2n} + 4(5n)^{2n} \quad (n > r/2).$$

Taking $2n$-th roots, and using the fact that $(a+b)^{1/2n} \le a^{1/2n} + b^{1/2n}$, valid for all $a, b > 0$, gives

$$M_{2n}^{1/2n} \le 4^{1/2n}|Z_{2n}|^{1/2n} + 4n + 4^{1/2n}5n \le 4|Z_{2n}|^{1/2n} + 24n \quad (n > r/2).$$

After dividing through by $M_{2n}^{1/2n}|Z_{2n}|^{1/2n}$, this becomes

$$|Z_{2n}|^{-1/2n} \le 4M_{2n}^{-1/2n} + 24nM_{2n}^{-1/2n}|Z_{2n}|^{-1/2n} \quad (n > r/2).$$

By (9), there exists an integer n_0 such that $24nM_{2n}^{-1/2n} < \frac{1}{2}$ for all $n > n_0$. Therefore

$$\frac{1}{2}|Z_{2n}|^{-1/2n} \le 4M_{2n}^{-1/2n} \quad \big(n > \max(r/2, n_0)\big).$$

Thus from the supposition $\sum_{n=1}^{\infty} M_n^{-1/n} < \infty$, it follows that $\sum_{n=1}^{\infty} |Z_{2n}|^{-1/2n} < \infty$, contrary the hypothesis (6) of the theorem. So finally we conclude that $\sum_{n=1}^{\infty} M_n^{-1/n} = \infty$, as claimed, and the proof of the theorem is complete. ∎

5. A quantitative theorem

This section is devoted to the following quantitative version of Theorem 1.1.

THEOREM 5.1: *Let $(a_n)_{n\ge 0}$ be a sequence of complex numbers, and for $n \ge 0$ set*

$$u_n = \sum_{\substack{k=0\\ k \text{ even}}}^{n} \binom{n}{k} a_k \quad \text{and} \quad v_n = \sum_{\substack{k=0\\ k \text{ odd}}}^{n} \binom{n}{k} a_k.$$

Suppose that there exist a real number $\beta \ge 1$ and an integer $r \ge 0$ such that, for all $n \ge 0$,

$$|u_n| \le \beta^n(n+1)^r \quad \text{and} \quad |v_n| \le \beta^n(n+1)^r. \tag{10}$$

Then, for all $n \ge r$,

$$|a_n| \le \begin{cases} C(0,\beta)\alpha^n\big(1+\log(n+1)\big), & \text{if } r = 0, \\ C(r,\beta)\alpha^{n-r}(n+1)^r, & \text{if } r \ge 1, \end{cases} \tag{11}$$

where

$$\alpha = \sqrt{\beta^2 - 1} \quad \text{and} \quad C(r,\beta) = r!(\sqrt{2}e)(1 + 1/\sqrt{2})^{r+1}\beta^{2r+1}. \tag{12}$$

In particular, taking $\beta = 1$, we have $\alpha = 0$, and thus we recover Theorem 1.1 as a special case (Theorem 5.1 is expressed in terms of powers of $n+1$ rather than of n as a matter of convenience, since this avoids the necessity of making separate statements for $n = 0$). The proof of Theorem 5.1 is based on Cauchy's integral formula, so in fact it provides a second proof of Theorem 1.1, independent of the theory of entire functions.

More generally, no matter what the value of β, Theorem 5.1 shows that, starting from the estimates (10) for u_n, v_n, one always obtains an analogous estimate (11) for a_n, but with β replaced by the 'improved' value $\alpha = \sqrt{\beta^2 - 1}$. There remains an exceptional case $r = 0$, where an extra $\log(n + 1)$ appears in (11). We do not know if this is really necessary.

That the value α given in (12) is sharp is easily seen from the following example. Fix $r \geq 0$, $\beta \geq 1$, and set $\gamma = \sqrt{\beta^2 - 1}$. For $n \geq 0$ set

$$a_n = n(n-1)\cdots(n-r+1)(i\gamma)^n.$$

Then

$$\begin{aligned} u_n &= \operatorname{Re}\sum_{k=0}^{n}\binom{n}{k}k(k-1)\cdots(k-r+1)(i\gamma)^k \\ &= \operatorname{Re} n(n-1)\cdots(n-r+1)(1+i\gamma)^{n-r}, \end{aligned}$$

with an analogous expression for v_n using imaginary parts. Thus (10) holds. On the other hand, it is clear that for $n \geq r$ we have

$$|a_n| \geq \text{constant} \times \gamma^{n-r}(n+1)^r.$$

Thus, if (11) is to hold for all $n \geq r$, we must have $\alpha \geq \gamma = \sqrt{\beta^2 - 1}$.

The other constant $C(r, \beta)$ in (11) is of lesser importance, and the estimate given in (12) is probably not optimal. However, there is a simple example which at least explains the presence of the principal term, namely $r!$. Fix $r \geq 0$, $\beta \geq 1$, and set

$$a_n = \begin{cases} \beta^r r! & \text{if } n = r, \\ 0 & \text{if } n \neq r. \end{cases}$$

Then, evidently $u_n = v_n = 0$ for all $n < r$, while for $n \geq r$ we have

$$|u_n|, |v_n| \leq \binom{n}{r}\beta^r r! \leq \beta^n (n+1)^r.$$

Thus (10) holds for all n. If (11) is to be valid for all $n \geq r$, then in particular it holds for $n = r$, and so we must have $C(r, \beta) \geq \beta^r r!$.

For the proof of Theorem 5.1, we shall need the following simple estimate.

LEMMA 5.2: *Let m, r be integers with $m \geq 1$ and $r \geq 0$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\sin m\theta}{\sin\theta}\right|^{r+1} d\theta \leq \begin{cases} 1+\log m, & \text{if } r=0, \\ m^r, & \text{if } r \geq 1. \end{cases}$$

Proof: Note first that

$$\frac{\sin m\theta}{\sin\theta} = e^{-i(m-1)\theta} + e^{-i(m-3)\theta} + \cdots + e^{i(m-1)\theta}.$$

In particular, it follows that, for each $\theta \in [0, 2\pi]$,

$$\left|\frac{\sin m\theta}{\sin\theta}\right| \leq m.$$

We start with the case $r = 0$. In this case,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\sin m\theta}{\sin\theta}\right| d\theta &= \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{\sin m\theta}{\sin\theta}\right| d\theta \\
&= \int_0^1 \left|\frac{\sin(m\pi t/2)}{\sin(\pi t/2)}\right| dt \\
&\leq \int_0^{1/m} m\,dt + \int_{1/m}^1 \frac{1}{t}\,dt \\
&= 1 + \log m.
\end{aligned}$$

Next, for the case $r = 1$, we observe that

$$\begin{aligned}
\left(\frac{\sin m\theta}{\sin\theta}\right)^2 &= (e^{-i(m-1)\theta} + e^{-i(m-3))\theta} + \cdots + e^{i(m-1)\theta})^2 \\
&= e^{-2i(m-1)\theta} + 2e^{-2i(m-2)\theta} + \cdots + m + \cdots + e^{2i(m-1)\theta},
\end{aligned}$$

and thus we actually have the equality

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\sin m\theta}{\sin\theta}\right)^2 d\theta = m.$$

Finally, the case $r \geq 2$ follows by remarking that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\sin m\theta}{\sin\theta}\right|^{r+1} d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\sin m\theta}{\sin\theta}\right)^2 m^{r-1}\,d\theta = mm^{r-1} = m^r. \qquad \blacksquare$$

Proof of Theorem 5.1: The even values of n interact with the odd values neither in the hypothesis (10) nor in the conclusion (11). The two can thus be treated separately, and there is no loss of generality in supposing, for example, that $a_n = 0$ for all odd n. In this case we have $v_n = 0$ for all n.

For $z \in \mathbf{C}$, define

$$a(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n, \quad b(z) = \sum_{n=0}^{\infty} \frac{u_n}{n!} z^n \quad \text{and} \quad c(z) = \sum_{n=0}^{\infty} \frac{(-1)^n u_n}{n!} z^n.$$

As in the proof of Theorem 1.1, a, b, c are entire functions of exponential type at most β, and

$$a(z) = e^{-z} b(z) = e^{z} c(z) \quad (z \in \mathbf{C}).$$

Also, as before, their respective Laplace transforms A, B, C are holomorphic on $\operatorname{Re}\zeta > \beta$, and on this half-plane satisfy

$$A(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{\zeta^{n+1}}, \quad B(\zeta) = \sum_{n=0}^{\infty} \frac{u_n}{\zeta^{n+1}}, \quad C(\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n u_n}{\zeta^{n+1}}$$

and

$$A(\zeta) = B(\zeta + 1) = C(\zeta - 1).$$

Thus, B, C extend holomorphically to $\{|\zeta| > \beta\}$, and hence A extends holomorphically to $\{|\zeta+1| > \beta\} \cup \{|\zeta-1| > \beta\}$. In particular, A is holomorphic $\{|\zeta| > \alpha\}$, where $\alpha = \sqrt{\beta^2 - 1}$. This implies straightaway that

$$\limsup_{n\to\infty} |a_n|^{1/n} \le \alpha. \tag{13}$$

Our aim now is to improve this estimate.

Fix $n \ge 0$, and let m be the smallest integer such that $m > n/2$. Take also $\sigma > \beta$, and set $\rho = \sqrt{\sigma^2 - 1}$. Consider the integral

$$I = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \left(1 - \left(\frac{i\rho}{\zeta}\right)^{2m}\right)^{r+1} A(\zeta) \zeta^n \, d\zeta.$$

On the one hand, substituting into I the formula $A(\zeta) = \sum_{k=0}^{\infty} a_k/\zeta^{k+1}$ and integrating term by term (which is legitimate, by (13), since $\rho > \alpha$), we find that

$$I = \sum_{k=0}^{\infty} \frac{a_k}{2\pi i} \int_{|\zeta|=\rho} \left(1 - \left(\frac{i\rho}{\zeta}\right)^{2m}\right)^{r+1} \zeta^{n-k-1} \, d\zeta = a_n.$$

On the other hand, we can estimate $|I|$ by

$$|I| \le \frac{1}{2\pi} \int_{|\zeta|=\rho} \left|1 - \left(\frac{i\rho}{\zeta}\right)^{2m}\right|^{r+1} |A(\zeta)| |\zeta|^n \, |d\zeta|.$$

Now, writing $\zeta = i\rho e^{i\theta}$, we have

$$\Bigl|1 - \Bigl(\frac{i\rho}{\zeta}\Bigr)^{2m}\Bigr|^{r+1} = |1 - e^{-2im\theta}|^{r+1} = |2\sin m\theta|^{r+1}.$$

Also, if $|\zeta| = \rho$ and $\operatorname{Re}\zeta > 0$, then

$$\begin{aligned}|A(\zeta)| = |B(\zeta+1)| &\le \sum_{k=0}^{\infty} \frac{|u_k|}{|\zeta+1|^{k+1}} \le \sum_{k=0}^{\infty} \frac{\beta^k (k+1)^r}{|\zeta+1|^{k+1}} \\ &\le \sum_{k=0}^{\infty} \frac{\sigma^k (k+1)\cdots(k+r)}{|\zeta+1|^{k+1}} = \frac{r!}{|\zeta+1|(1-\sigma/|\zeta+1|)^{r+1}} \\ &= \frac{r!|\zeta+1|^r}{(|\zeta+1|-\sigma)^{r+1}} = \frac{r!|\zeta+1|^r(|\zeta+1|+\sigma)^{r+1}}{(|\zeta+1|^2-\sigma^2)^{r+1}}.\end{aligned}$$

Writing $\zeta = i\rho e^{i\theta}$ again, we have

$$|\zeta+1|^2 = 1 + \rho^2 + 2\rho|\sin\theta| \le 2(1+\rho^2) = 2\sigma^2$$

and

$$|\zeta+1|^2 - \sigma^2 = 1 + \rho^2 + 2\rho|\sin\theta| - \sigma^2 = |2\rho\sin\theta|.$$

Hence

$$|A(\zeta)| \le \frac{r!(\sqrt{2}\sigma)^r(\sqrt{2}\sigma+\sigma)^{r+1}}{|2\rho\sin\theta|^{r+1}} = \frac{r!(\sqrt{2})^r(\sqrt{2}+1)^{r+1}\sigma^{2r+1}}{|2\rho\sin\theta|^{r+1}}.$$

We obtain the same estimate for $|\zeta| = \rho$ and $\operatorname{Re}\zeta < 0$ by repeating the argument with $B(\zeta)$ replaced by $C(\zeta)$. Combining all these estimates, we deduce that

$$\begin{aligned}|I| &\le \frac{1}{2\pi}\int_0^{2\pi} |2\sin m\theta|^{r+1}\Bigl(\frac{r!(\sqrt{2})^r(\sqrt{2}+1)^{r+1}\sigma^{2r+1}}{|2\rho\sin\theta|^{r+1}}\Bigr)\rho^n \rho\, d\theta \\ &= r!(\sqrt{2})^r(\sqrt{2}+1)^{r+1}\sigma^{2r+1}\rho^{n-r}\frac{1}{2\pi}\int_0^{2\pi}\Bigl|\frac{\sin m\theta}{\sin\theta}\Bigr|^{r+1} d\theta.\end{aligned}$$

If we let $\sigma \to \beta$, then $\rho \to \alpha$, and thus, for $n \ge r$,

$$|a_n| \le r!(\sqrt{2})^r(\sqrt{2}+1)^{r+1}\beta^{2r+1}\alpha^{n-r}\frac{1}{2\pi}\int_0^{2\pi}\Bigl|\frac{\sin m\theta}{\sin\theta}\Bigr|^{r+1} d\theta.$$

Now, by Lemma 5.2, for $n \ge r \ge 1$ we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\Bigl|\frac{\sin m\theta}{\sin\theta}\Bigr|^{r+1} d\theta \le m^r \le \Bigl(\frac{n+2}{2}\Bigr)^r &\le 2^{-r}(n+1)^r\Bigl(\frac{n+2}{n+1}\Bigr)^r \\ &\le 2^{-r}(n+1)^r(1+1/r)^r \le 2^{-r}(n+1)^r e,\end{aligned}$$

whilst if $r = 0$ then, for all $n \ge 0$,

$$\frac{1}{2\pi}\int_0^{2\pi}\Bigl|\frac{\sin m\theta}{\sin\theta}\Bigr| d\theta \le 1 + \log m \le 1 + \log\Bigl(\frac{n+2}{2}\Bigr) \le 1 + \log(n+1).$$

Substituting these estimates into the preceding inequality for $|a_n|$, we finally obtain (11). This completes the proof. ∎

6. Concluding remarks and questions

(i) Theorem 1.1 remains true if we replace (1) by

$$\sum_{\substack{k=0 \\ k\equiv l \ (\mathrm{mod}\ q)}}^{n} \binom{n}{k} a_k = O(n^r) \quad (l = 1, \ldots, q),$$

provided that q is an integer with $q \geq 3$ (note that (1) is simply the case $q = 2$). The proof is essentially the same as that in §2, but now using a (slightly easier) variant of the Phragmén–Lindelöf principle, where one assumes polynomial growth of f along the half-lines $\arg(z) = 2\pi l/q$ $(l = 1, \ldots, q)$. We omit the details.

There is likewise an analogue of Theorem 5.1 for $q \geq 3$, in which α is now the smallest radius such that

$$\{\zeta\colon |\zeta| > \alpha\} \subset \bigcup_{l=1}^{q} \{\zeta\colon |\zeta - e^{2\pi i l/q}| > \beta\},$$

namely,

$$\alpha = \sqrt{\beta^2 - \sin^2(\pi/q)} - \cos(\pi/q).$$

Notice that $\alpha \to (\beta - 1)$ as $q \to \infty$.

(ii) The result mentioned in (i) above leads to an analogue of Theorem 3.2 for each $q \geq 3$, i.e. a sufficient condition for T^q to have a non-trivial closed invariant subspace. This raises a question. Do the hypotheses of Theorem 3.2 imply the existence of a non-trivial closed invariant subspace, not just for T^2, but for T itself?

(iii) Is the presence of $\log(n + 1)$ in (11) really necessary? We suspect not. Perhaps it could be eliminated by a judicious application of the theory of singular integrals.

(iv) Theorem 5.1 treats rates of growth of the form $\beta^n(n+1)^r$. However, there are other intermediate rates of growth which are also of interest. For example, in view of the work of Atzmon [2], it is natural to ask what one can conclude about (a_n) from the hypothesis that $u_n, v_n = O(e^{\sqrt{n}})$. The first part of the proof of Theorem 5.1 (up to (13)) shows that, necessarily, $|a_n|^{1/n} \to 0$. Is it possible to say more?

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